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## *The Apparent Size of a Closed Curve.*

BY ARTHUR C. LUNN.

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1. The apparent size of a closed curve in space, or the solid angle subtended by it as seen from any point, is defined as the solid angle of a cone with vertex at the point of sight and with the curve as directrix. This is measured, according to the definition of Gauss, by the area cut by the cone from a sphere of unit radius with center at the given point. It is equivalent to the apparent size of any surface bounded by the curve, provided a suitable convention be adhered to as to the algebraic sign of each element, according to whether one side or the other is visible from the point. Its value is then given by the familiar integral

$$\omega = - \int \frac{\cos(n, r) dS}{r^2} = - \int \frac{x \cos(n, x) + y \cos(n, y) + z \cos(n, z)}{r^3} dS \quad (1)$$

extended over a surface bounded by the curve, with positive side at each point fixed by the direction of the normal  $n$ . Here the point of sight is taken as origin of coordinates and an element of solid angle is considered positive when the origin is on the positive side of the corresponding element of surface,  $r$  being the radius vector from the origin to a point of the surface. The explicit computation of  $\omega$  is required, for example, in the determination of the magnetic potential of a closed linear electric current.\*

The integral in (1) may be evaluated as the sum of three double integrals, extended over the areas bounded by the projections of the curve on the coordinate planes, or as one double integral extended over a plane area into which the surface is mapped by means of a representation of the coordinates of its points in terms of two independent parameters. But inasmuch as the surface here is purely auxiliary and is arbitrary except in that it is to be bounded by the given curve, it seems more natural to seek a method of computation such as to use the curve itself as realm of integration, for instance by integration with respect to a parameter  $t$  in terms of which the coordinates of the points of the curve may be expressed.

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\* Gauss: "Allgemeine Theorie des Erdmagnetismus," *Gesammelte Werke*, V, p. 170.

One such method is given by the ordinary formula

$$\omega = \int (1 - \cos \phi) d\theta = \int (1 - \cos \phi) \theta' dt, \quad (2)$$

where  $\theta, \phi$  are the longitude and colatitude of the point where the unit sphere is intersected by the vector  $\mathbf{r}$  from the origin to a point of the curve. Here the use of  $t$  as a variable of integration, in terms of which the coordinates may be considered as single-valued, relieves the ambiguity as to the interval of variation of  $\theta$  arising from the fact that the polar axis, being arbitrary, may or may not be linked with the curve. For comparison with other formulas to be considered it will be convenient to throw this into a form containing rectangular coordinates, which will here be done by means of a vector formula independent of any particular coordinate system.

Let  $\mathbf{a}$  be a constant vector, of length  $a$ , in the direction of the polar axis implied in equation (2), and let  $\mathbf{r}_1, \mathbf{a}_1$  be unit vectors in the directions of  $\mathbf{r}$  and  $\mathbf{a}$  respectively, so that  $\mathbf{r} = r\mathbf{r}_1, \mathbf{a} = a\mathbf{a}_1$ . Then if  $t$  be thought of as time,  $\theta'$  is the angular velocity of the plane of the vectors  $\mathbf{r}, \mathbf{a}$  about the fixed vector  $\mathbf{a}$  as axis; and  $\cos \phi = \mathbf{r}_1 \cdot \mathbf{a}_1$ .<sup>\*</sup> Now a vector normal to the plane in question is  $\mathbf{n} = n\mathbf{n}_1 = \mathbf{r} \times \mathbf{a}$ , so that the derivative  $\mathbf{n}'$  of the unit normal vector is numerically equal to  $\theta'$ . From this come:

$$\begin{aligned} \mathbf{n}' &= n\mathbf{n}'_1 + n'\mathbf{n}_1 = \mathbf{r}' \times \mathbf{a}, \quad n^2 = (\mathbf{r} \times \mathbf{a})^2, \quad n n' = (\mathbf{r}' \times \mathbf{a}) \cdot (\mathbf{r} \times \mathbf{a}), \\ n^2 \mathbf{n}'_1 &= n(\mathbf{r}' \times \mathbf{a}) - \mathbf{n}_1 (\mathbf{r}' \times \mathbf{a}) \cdot (\mathbf{r} \times \mathbf{a}), \end{aligned}$$

and from the last, by squaring and reducing:

$$n^4 \mathbf{n}'_1{}^2 = \{(\mathbf{r} \times \mathbf{a}) \times (\mathbf{r}' \times \mathbf{a})\}^2 = \{[\mathbf{r}' \mathbf{r} \mathbf{a}] \mathbf{a}\}^2, \quad (3)$$

which shows that the vector of angular velocity of the plane may be taken as

$$\mathbf{u} = \frac{[\mathbf{r}' \mathbf{r} \mathbf{a}]}{(\mathbf{r} \times \mathbf{a})^2} (-\mathbf{a}), \quad (4)$$

which is numerically equal to  $\mathbf{n}'_1$ , but is parallel to the axis  $\mathbf{a}$  while  $\mathbf{n}'_1$  is perpendicular. This formula may be obtained also directly by a geometric construction, with  $\mathbf{r}'$  treated as the velocity of a point moving along the curve, so that  $-[\mathbf{r}' \mathbf{r} \mathbf{a}]/a$  is the moment of the velocity with respect to the axis. The sense of  $\mathbf{u}$ , left undetermined by (3), is here so chosen that when  $\mathbf{r}', \mathbf{r}, \mathbf{a}$  form a right-handed triple of vectors the direction of  $\mathbf{u}$  shall be opposite to that of  $\mathbf{a}$ . Thus if the forward circuit of the curve appear counter-clockwise as seen from

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<sup>\*</sup> The vector notation is throughout that of Gibbs-Wilson: "Vector Analysis."

what is taken in (1) as the positive side of the surface, the resulting value of the solid angle is

$$\omega = \int \left( 1 - \frac{\mathbf{r} \cdot \mathbf{a}}{r a} \right) \frac{[\mathbf{r}' \mathbf{r} \mathbf{a}] a}{(\mathbf{r} \times \mathbf{a})^2} dt, \quad (5)$$

which in Cartesian rectangular coordinates is

$$\left. \begin{aligned} \omega &= \int \left( 1 - \frac{a_1 x + a_2 y + a_3 z}{a r} \right) \frac{a}{n^2} \begin{vmatrix} x' & y' & z' \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} dt, \\ r &= \sqrt{x^2 + y^2 + z^2}, \quad a = \sqrt{a_1^2 + a_2^2 + a_3^2}, \\ n^2 &= (a_2 z - a_3 y)^2 + (a_3 x - a_1 z)^2 + (a_1 y - a_2 x)^2 \\ &= a^2 r^2 - (a_1 x + a_2 y + a_3 z)^2, \end{aligned} \right\} \quad (6)$$

where  $a_1, a_2, a_3$  are the scalar components of  $\mathbf{a}$ . In this deduction the polar axis has any arbitrary direction with respect to the rectangular axes; if it be taken to coincide with the  $z$ -axis, and  $t$  be the arc of the curve, this formula reduces to one used by Maxwell.\*

The integrand in this formula is independent of the particular set of axes used for the resolution of the vectors; and the integral is independent of the special choice of the parameter  $t$ , but it contains the direction of the arbitrary vector  $\mathbf{a}$  and thus depends apparently on the ratios of the scalar constants  $a_1, a_2, a_3$ . The geometric source of the formula shows, however, that the integral around a closed curve must be independent of those constants, for any continuous variation of them which does not introduce singularities into the integrand.

If the curve intersect the axial line, then at such a point  $\mathbf{r}$  becomes parallel to  $\mathbf{a}$  and the divisor  $n^2$  vanishes. Since  $\mathbf{a}$  is arbitrary, this could of course be avoided for any particular curve, but there is a feature of theoretical interest in such a case.

If the intersection is on the positive half of the axis of  $\mathbf{a}$ , so that  $\phi = 0$ , the vanishing of other factors keeps the integrand finite; but if it is on the negative half, with  $\phi = \pi$ , the integrand becomes infinite. In the latter case a suitable value of  $\omega$  might be sought by a limiting process, for which purpose a short arc of the curve in the neighborhood of the intersection is replaced by a circular arc, for instance, with center on the axis. But the result then depends upon which way this arc lies with respect to the axis, there being in fact, for

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\* "Electricity and Magnetism," Vol. II, Ed. 3, p. 39.

arcs on opposite sides of the axis, two possible limiting values of the integral differing by  $4\pi$ , which is the limiting value of the integral in (2) or (5) for a small circle with center on the negative part of the axis. Thus a variation of the vector  $\mathbf{a}$  such as to make its negative continuation pass across the curve produces an abrupt change of  $4\pi$  in the value of the integral. The obvious interpretation is that this  $4\pi$  is the apparent size of the whole "sky," or the cyclic constant which occurs additively an arbitrary integral number of times in the general many-valued determination of  $\omega$ . Moreover, if two different expressions for the integrand in (6) be formed, corresponding to different choices of the vector  $\mathbf{a}$ , their difference proves to be the exact derivative of a certain function of the coordinates which is expressible in terms of inverse trigonometric tangents and is many-valued with the same cyclic constant  $4\pi$ .

2. Another method of computation\* depends on the relation  $\omega = 2\pi - \sigma$ , generalized from a familiar theorem relative to polar spherical triangles, in which  $\sigma$  is the length of that curve on the unit sphere which is polar to the intersection of sphere and cone. To translate this into an explicit general formula, let it be noticed that if the vector  $\mathbf{r}$  goes to a point of the given curve then the unit vector of the vector  $\mathbf{p} = \mathbf{r} \times \mathbf{r}'$  goes to the corresponding point of the polar curve on the sphere. Then

$$p \mathbf{p}' = \mathbf{r} \times \mathbf{r}'' - p' \mathbf{p}_1, \quad p^2 = (\mathbf{r} \times \mathbf{r}')^2, \quad p p' = (\mathbf{r} \times \mathbf{r}') \cdot (\mathbf{r} \times \mathbf{r}''),$$

and from the first of these, by squaring and reducing:

$$p^4 \mathbf{p}_1'^2 = r^2 [\mathbf{r}'', \mathbf{r}', \mathbf{r}]^2. \quad (7)$$

Here  $\mathbf{p}_1'$  is numerically the derivative of the arc of the polar curve, so that with a conventional choice of sign in taking the square root in (7) the solid angle is

$$\omega = 2\pi - \int \frac{r [\mathbf{r}'', \mathbf{r}', \mathbf{r}]}{(\mathbf{r} \times \mathbf{r}')^2} dt, \quad (8)$$

or in Cartesian coordinates:

$$\left. \begin{aligned} \omega &= 2\pi - \int \frac{r}{p^2} \begin{vmatrix} x'' & y'' & z'' \\ x' & y' & z' \\ x & y & z \end{vmatrix} dt, \\ p^2 &= (y z' - z y')^2 + (z x' - x z')^2 + (x y' - y x')^2 \\ &= r^2 q^2 - (x x' + y y' + z z')^2, \\ q^2 &= x'^2 + y'^2 + z'^2. \end{aligned} \right\} \quad (9)$$

\* Used for example by Schwartz: *Gesammelte Werke*, I, p. 316.

To put this into a form using polar coordinates, for comparison with (2), let  $(x, y, z)$  be written  $(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi)$ ; then it reduces to

$$\omega = 2\pi - \int \frac{(\phi'' \theta' - \theta'' \phi') \sin \phi - 2 \theta' \phi'^2 \cos \phi - \theta'^3 \sin^2 \phi \cos \phi}{\phi'^2 + \theta'^2 \sin^2 \phi} dt. \quad (10)$$

The divisor  $p^2$  vanishes at a point of the curve where the tangent line coincides with the radius vector. At such a point the integrand may remain finite if also the osculating plane pass through the origin; but in general it becomes infinite and the value of  $\omega$  needs to be found by a limiting process, involving a system of curves obtained by warping the original so as to keep  $p$  different from zero. The typical case of such a critical point is that where the curve is such that its trace on the unit sphere presents a cusp. Then, according to whether the curve is deformed so as to change this cusp to a rounded arc or to some kind of loop, the limiting process gives a result indeterminate by an additive multiple of  $4\pi$ , like that mentioned in § 1.

In contrast with the formula of § 1, the present one contains no arbitrary constants, but it involves the second derivatives of the coordinates and thus implies stronger conditions on the curve. For example, equation (5) or (6) could be used directly for a polygon or other curve having in general a definite tangent but with a suitably restricted set of points of discontinuity in the direction of that tangent; while in such a case equation (8) or (9) would be inapplicable unless for each such point there be added a corrective term measured by the corresponding change in direction of the line tangent to the trace on the unit sphere.

3. The method now to be developed consists in a transformation of the surface integral of equation (1) into a line integral by means of Stokes's theorem:

$$\int \text{curl } \mathbf{v} \cdot d\mathbf{a} = \int \mathbf{v} \cdot d\mathbf{r}, \quad (11)$$

where  $d\mathbf{a}$  is the vector equivalent of the scalar element of area  $dS$ .

If  $\mathbf{R}$  be put for the Newtonian vector  $-\mathbf{r}_1/r^2$ , then equation (1) may be written

$$\omega = \int \mathbf{R} \cdot d\mathbf{a}. \quad (12)$$

Thus if there is a vector function  $\mathbf{v}$  such that  $\text{curl } \mathbf{v} = \mathbf{R}$ , then under the analytic conditions sufficient for Stokes's theorem the solid angle can be computed from

$$\omega = \int \mathbf{v} \cdot \mathbf{r}' dt. \quad (13)$$

Since  $\mathbf{R}$  is a solenoidal vector function, there must exist for it such a vector potential  $\mathbf{v}$ , whose determination in terms of rectangular coordinates involves the integration of the system of differential equations:

$$\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} = -\frac{x}{r^3}, \quad \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = -\frac{y}{r^3}, \quad \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = -\frac{z}{r^3}, \quad (14)$$

which condition its scalar components  $(v_1, v_2, v_3)$ .

By these equations alone  $\mathbf{v}$  is not uniquely determined, but any two solutions differ by a potential vector  $\nabla U$  or one whose components are partial derivatives of a single scalar function  $U$ . This  $U$  may be many-valued, but its discontinuities and those of  $\nabla U$  must be allowed for by corresponding corrective terms in the application of Stokes's theorem. It may be predicted that the only indeterminateness thus introduced will be such that the values of  $\omega$  computed by means of two different expressions for  $\mathbf{v}$  will differ by a multiple of  $4\pi$ . A general proof of this is not attempted here, but a verification is given below for some particular cases.\*

A particular solution of equations (14), found by a known tentative process, is the following:

$$\left. \begin{aligned} v_1 &= \frac{1}{3r} \left( \frac{y}{r+z} - \frac{z}{r+y} \right), \\ v_2 &= \frac{1}{3r} \left( \frac{z}{r+x} - \frac{x}{r+z} \right), \\ v_3 &= \frac{1}{3r} \left( \frac{x}{r+y} - \frac{y}{r+x} \right), \end{aligned} \right\} \quad (15)$$

which may be written

$$\mathbf{v} = \frac{1}{3} \mathbf{r}_1 \times \mathbf{w}; \quad \mathbf{w} = \left( \frac{1}{r+x}, \frac{1}{r+y}, \frac{1}{r+z} \right). \quad (16)$$

Then the general solution may be written  $\mathbf{v} + \nabla U$ , where the scalar function  $U$  is arbitrary. The special form of  $\mathbf{v}$  just found gives, by substitution in (13):

$$\omega = \frac{1}{3} \int [\mathbf{r}', \mathbf{r}_1, \mathbf{w}] dt, \quad (17)$$

which in Cartesian coordinates is

$$\omega = \frac{1}{3} \int \begin{vmatrix} x' & y' & z' \\ x & y & z \\ \frac{1}{r+x} & \frac{1}{r+y} & \frac{1}{r+z} \end{vmatrix} dt. \quad (18)$$

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\* For a theory of many-valued functions in a Riemann space see a paper by Dixon, *Proceedings of the London Mathematical Society*, Ser. 2, Vol. I, pp. 415-436.

With respect to this formula a qualification must be made, however, to cover cases where the vanishing of any of the quantities  $(r + x, r + y, r + z)$  makes  $\mathbf{v}$  infinite at any point involved in the integrals of equation (11). Since continuity of  $\mathbf{v}$  and of  $\text{curl } \mathbf{v}$  is presupposed in Stokes's theorem, a corrective term is needed for every point where the negative part of a coordinate axis pierces the surface implied in (11), even when at all points of the curve itself the integrand in (18) remains finite.

To find the magnitude of this correction it suffices to consider a special case, such as a circle with center on the negative  $z$ -axis and lying in a normal plane, represented by

$$x = \rho \cos t, \quad y = \rho \sin t, \quad z = -b,$$

where  $\rho, b$  are positive constants, so that  $r$  is the constant  $\sqrt{\rho^2 + b^2}$ . For the integral in (18) this gives

$$\frac{1}{3r} \int_0^{2\pi} dt \left\{ -\frac{\rho^2}{r-b} - \frac{b\rho \sin t}{r+\rho \sin t} - \frac{b\rho \cos t}{r+\rho \cos t} \right\} = 2\pi \left( \frac{1}{3} - \frac{b}{r} \right),$$

while the true value for  $\omega$  is  $2\pi(1 - b/r)$ . Hence it may be inferred in general that for every point where the negative part of a coordinate axis, when taken in the direction from the origin to infinity, pierces the surface from its positive or negative side, a corresponding correction  $4\pi/3$ , positive or negative respectively, must be added to the integral in (18). Thus the passage of the curve across a negative semi-axis during continuous deformation would be marked by a discontinuity of  $4\pi/3$  in the value of the integral.

For example, suppose  $\omega_0$  is the value of the integral for a curve linked with  $k$  of the negative semi-axes. By reversal of the sense of integration the same curve may be considered to define also the complementary solid angle with  $3-k$  corrective terms. These mutually complementary solid angles are then  $\omega_0 + 4\pi k/3$  and  $-\omega_0 + 4\pi(3-k)/3$ , whose sum is  $4\pi$ , as it should be.

If two curves lie on the same cone, with vertex at the point of sight, and are represented in terms of the same parameter  $t$  by a correspondence of those points which lie on the same linear element of the cone, then if the vector to one curve be  $\mathbf{r}$ , the corresponding vector to the other curve is  $\phi \mathbf{r}$ , where  $\phi$  is a scalar function of  $t$ . On account of homogeneity of degree zero in the coordinates, the integrand in (17) is therefore the same for both curves, since the derivative  $\phi'$  has as coefficient a vanishing determinant. A similar remark applies to the alternative formulas (5) and (8).

Another solution of equations (14) is that obtained from (15) by reversing the signs of the coordinates, but not of  $r$ , and then changing the sign of the components of  $\mathbf{v}$  to allow for the resulting interchange of right-handed and left-handed reckoning of rotations. The corresponding value of the solid angle in Cartesian form is

$$\omega = -\frac{1}{3} \int \left| \begin{array}{ccc} x' & y' & z' \\ x & y & z \\ 1 & 1 & 1 \end{array} \right| \frac{dt}{r}, \quad (19)$$

which is to be subjected to a correction of  $4\pi/3$  for each piercing of the surface by a positive semi-axis. The value of  $\mathbf{v}$  here used is obtainable from the former by addition of the vector whose components are partial derivatives of the scalar function

$$U = \frac{2}{3} \left\{ \tan^{-1} \frac{z}{y} + \tan^{-1} \frac{x}{z} + \tan^{-1} \frac{y}{x} \right\}.$$

Another form is the mean of the two:

$$\omega = -\frac{1}{3} \int \left| \begin{array}{ccc} x' & y' & z' \\ x & y & z \\ \frac{x}{y^2+z^2} & \frac{y}{z^2+x^2} & \frac{z}{x^2+y^2} \end{array} \right| \frac{dt}{r}, \quad (20)$$

which needs correction by  $2\pi/3$  for each piercing of the surface by either a positive or a negative semi-axis. This last form has the advantage that the irrationality  $r = \sqrt{x^2 + y^2 + z^2}$  does not occur in the determinant.

The differential equations (14) have the same form for all systems of rectangular axes, but any particular solution for the vector  $\mathbf{v}$  has components whose analytic form depends on the choice of axes. Thus if formula (18) be considered as written for one particular system of axes and then be transformed to another system, the resulting formula will contain the arbitrary constants which specify the relative orientation of the new system with respect to the old.

If  $(x_1, y_1, z_1)$  be coordinates referred to a system of axes obtained from the system of  $(x, y, z)$  by rigid rotation, as represented by the equations

$$\left. \begin{aligned} x_1 &= a_{11}x + a_{12}y + a_{13}z, \\ y_1 &= a_{21}x + a_{22}y + a_{23}z, \\ z_1 &= a_{31}x + a_{32}y + a_{33}z, \end{aligned} \right\} \quad (21)$$

with the familiar conditions on the coefficients, then the integrand, formed according to the scheme of (18), but in terms of  $(x_1, y_1, z_1)$  and then expressed in terms of  $(x, y, z)$ , is equivalent to

$$\frac{1}{r} \begin{vmatrix} x' & x & \frac{a_{11}}{r+x_1} + \frac{a_{21}}{r+y_1} + \frac{a_{31}}{r+z_1} \\ y' & y & \frac{a_{12}}{r+x_1} + \frac{a_{22}}{r+y_1} + \frac{a_{32}}{r+z_1} \\ z' & z & \frac{a_{13}}{r+x_1} + \frac{a_{23}}{r+y_1} + \frac{a_{33}}{r+z_1} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad (22)$$

where the second determinant factor is unity, and  $(x_1, y_1, z_1)$  are now to be understood merely as abbreviations. This may be obtained from the integrand explicitly given in (18) by addition of the exact derivative of the scalar function

$$U_1 = \frac{2}{3} \left\{ \begin{aligned} & \tan^{-1} \frac{(1-a_{11})z + a_{13}(r+x)}{(1-a_{11})y + a_{12}(r+x)} \\ & + \tan^{-1} \frac{(1-a_{22})x + a_{21}(r+y)}{(1-a_{22})z + a_{23}(r+y)} \\ & + \tan^{-1} \frac{(1-a_{33})y + a_{32}(r+z)}{(1-a_{33})x + a_{31}(r+z)} \end{aligned} \right\}.$$

Further alternative formulas for  $\omega$  could be obtained indefinitely by taking as integrand any arbitrarily weighted mean of expressions such as (22), each corresponding to a different transformation of axes, and demanding for the axes belonging to each term a similarly weighted correction for discontinuity. Of special interest among such would be those where the various systems of axes are related by means of the transformations of a certain rotation group, so that the resulting formula would be formally invariant under the transformations of that group, if each term of the integrand be given the same weight.